The maximum dimension of a Lie nilpotent subalgebra of $M_n(F)$ of index m

Michał Ziembowski

Warsaw University of Technology

Lens
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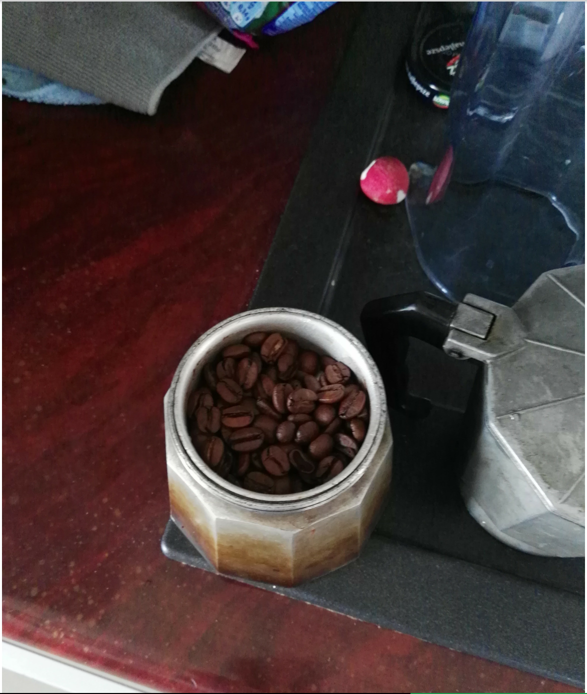
Based on a joint work with J. van den Berg, J. Szigeti and L. van Wyk



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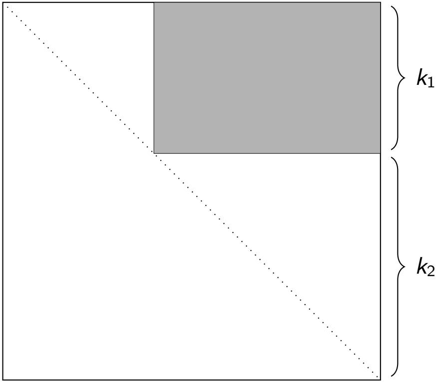
$J =$ 

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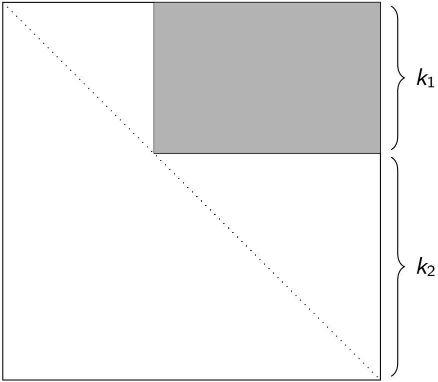
$J =$ 

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- (Schur 1905, Jacobson 1944) The dimension over a field F of any commutative subalgebra of $\mathbb{M}_n(F)$ is at most $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$, where $\lfloor\ \rfloor$ is the floor function.

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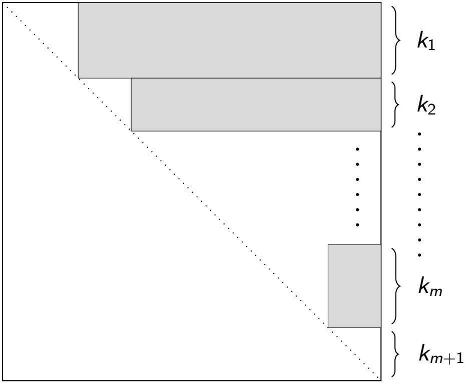
$J =$ 

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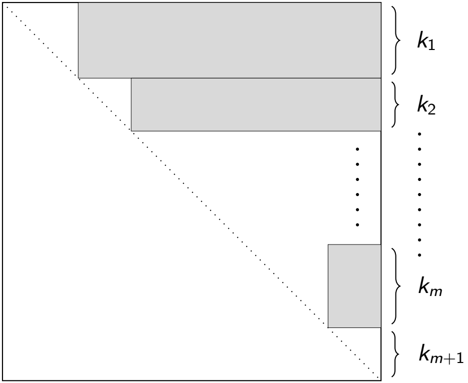
$J =$ 

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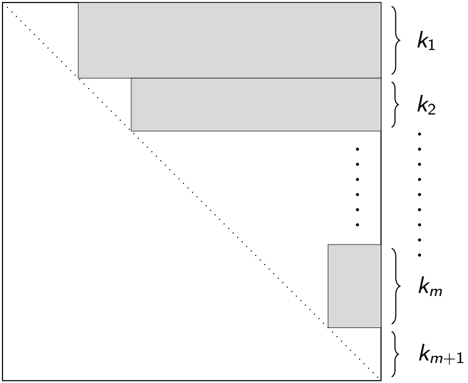

- Let $R = FI_n + J$ ("TYPICAL EXAMPLE")
- Obviously: $J(R) = J$, $J^{m+1} = 0$, and R is local.

- $$\begin{aligned}\dim_F R &= k_1(n-k_1)+k_2(n-k_1-k_2)+\cdots \\ &\quad +k_m(n-k_1-k_2-\cdots-k_m)+1 \\ &= \sum_{i,j=1,\, i<j}^{m+1} k_i k_j + 1.\end{aligned}$$

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  $$= \textstyle\sum_{i,j=1,\, i<j}^{m+1} k_i k_j + 1.$$
- $$M(\ell, n) \stackrel{\text{def}}{=} \max\left\{ \sum_{i,j=1,\, i<j}^{\ell} k_i k_j + 1 : k_1, k_2, \ldots, k_\ell \text{ are nonnegative integers such that } \sum_{i=1}^{\ell} k_i = n \right\}.$$

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- We get $M(\ell, n)$ for the sequence $(k_1, k_2, \dots, k_\ell) \in \mathbb{N}_0^\ell$ defined in the following way:

$$k_i \stackrel{\text{def}}{=} \begin{cases} \left\lfloor \frac{n}{\ell} \right\rfloor, & \text{for } 1 \leqslant i \leqslant \ell - r \\ \left\lfloor \frac{n}{\ell} \right\rfloor + 1, & \text{for } \ell - r < i \leqslant \ell. \end{cases}$$

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- The *Lower Central Series* $\{\mathfrak{L}_{[m]}\}_{m \in \mathbb{N}}$ of $\mathfrak{L}$ is defined by

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- Definition: We say $\mathfrak{L}$ is *nilpotent* if $\mathfrak{L}_{[m]} = 0$ for some $m \in \mathbb{N}$, $m > 1$, and more specifically, *nilpotent of index m*, if $\mathfrak{L}_{[m+1]} = 0$.

- Every ring R may be endowed with the structure of a Lie algebra (over the centre of R), by choosing as bracket the *commutator* defined by

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- "TYPICAL EXAMPLE" is an algebra which is Lie nilpotent of index m.

- **Conjecture.** *(J. Szigeti, L. van Wyk)* Let F be any field, m and n positive integers, and R an F-subalgebra of $\mathbb{M}_n(F)$ with Lie nilpotence index m. Then

$$\dim_F R \leqslant M(m+1, n).$$

Theorem 1. Let $\mathcal{C}$ be a nonempty class of fields and $\overline{\mathcal{C}}$ the class of all subfields of fields in $\mathcal{C}$. The following statements are equivalent:

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- The Conjecture holds in respect of F;
- For all positive integers m and n, if R is any F-subalgebra of $\mathbb{U}_n^*(F)$ (upper triangular $n \times n$ matrices over F with constant main diagonal) with Lie nilpotence index m, then

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- $$\begin{cases} R_1 \stackrel{\text{def}}{=} R, \\ J_1 \stackrel{\text{def}}{=} J(R_1), \text{ and} \\ U_1 \stackrel{\text{def}}{=} \text{ any } F\text{-subspace complement of } VJ_1 \text{ in } V. \end{cases}$$

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- For $k \in \mathbb{N}$, $k \geqslant 2$, define

$$\begin{cases} R_k \stackrel{\text{def}}{=} FI_n + (0 :^{R_{k-1}} U_{k-1}), \\ J_k \stackrel{\text{def}}{=} J(R_k), \text{ and} \\ U_k \stackrel{\text{def}}{=} \text{ any } F\text{-subspace complement of } VJ_1J_2\dots J_k \text{ in } \\ \qquad VJ_1J_2\dots J_{k-1}. \end{cases}$$

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- Theorem 3. Let the sequences $\{R_k\}_{k\in\mathbb{N}}$, $\{J_k\}_{k\in\mathbb{N}}$ and $\{U_k\}_{k\in\mathbb{N}}$ be defined as above, and let ℓ be as above. Then:

  (i) $\dim_F R \leqslant M(\ell, \dim_F V)$.
  (ii) If R is Lie nilpotent of index m, then $\ell \leqslant m+1$

Theorem 4. (J. van den Berg, J. Szigeti, L. van Wyk and M.Z.) *Let F be any field, m and n positive integers, and R an F-subalgebra of $\mathbb{M}_n(F)$ with Lie nilpotence index m. Then*

$$\dim_F R \leqslant M(m+1, n).$$

PROBLEM: Every ring R that is Lie nilpotent of index m, is also Lie solvable of index m. Thus, it is natural to ask about the maximal dimension of Lie solvable of index m subalgebras of $M_n(F)$.

Thank you for your attention!